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ON SUMS OF BINOMIAL COEFFICIENTS AND THEIR APPLICATIONS

Zhi-Wei Sun

Department of Mathematics, Nanjing University
Nanjing 210093, People's Republic of China
zwsun@nju.edu.cn
http://math.nju.edu.cn/~zwsun

Abstract. In this paper we study recurrences concerning the combinatorial sum $\left[{n \atop r}\right]_m = \sum_{k\equiv r \pmod m}\binom{n}{k}$ and the alternate sum $\sum_{k\equiv r \pmod m}(-1)^{(k-r)/m}\binom{n}{k}$, where $m>0$, $n\geqslant 0$ and r are integers. For example, we show that if $n\geqslant m-1$ then

$$\sum_{i=0}^{\lfloor (m-1)/2\rfloor}(-1)^i\binom{m-1-i}{i}\left[{n-2i \atop r-i}\right]_m = 2^{n-m+1}.$$

We also apply such results to investigate Bernoulli and Euler polynomials. Our approach depends heavily on an identity established by the author [*Integers* **2**(2002)].

Keywords: Binomial coefficient; combinatorial sum; recurrence; Bernoulli polynomial; Euler polynomial.

1. Introduction and Main Results

As usual, we let

$$\binom{x}{0}=1 \text{ and } \binom{x}{n}=\frac{x(x-1)\cdots(x-n+1)}{n!} \text{ for } n\in\mathbb{Z}^+=\{1,2,3,\ldots\}.$$

Following [Su2], for $m\in\mathbb{Z}^+$, $n\in\mathbb{N}=\{0,1,2,\ldots\}$ and $r\in\mathbb{Z}$ we set

$$(1.1)\qquad \left[{n \atop r}\right]_m = \sum_{\substack{k=0\\ k\equiv r \pmod m}}^{n}\binom{n}{k} \quad\text{and}\quad \left\{{n \atop r}\right\}_m = \sum_{\substack{k=0\\ k\equiv r \pmod m}}^{n}(-1)^{\frac{k-r}{m}}\binom{n}{k}.$$

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As $\binom{n+1}{k}=\binom{n}{k}+\binom{n}{k-1}$ for any $k\in\mathbb{Z}^+$, we have the following useful recursions:

$$\begin{bmatrix}n+1\\ r\end{bmatrix}_m=\begin{bmatrix}n\\ r\end{bmatrix}_m+\begin{bmatrix}n\\ r-1\end{bmatrix}_m \quad\text{and}\quad \left\{\begin{matrix}n+1\\ r\end{matrix}\right\}_m=\left\{\begin{matrix}n\\ r\end{matrix}\right\}_m+\left\{\begin{matrix}n\\ r-1\end{matrix}\right\}_m. \tag{1.2}$$

Let $m,n\in\mathbb{Z}^+$ and $r\in\mathbb{Z}$. The study of the sum $\left[\begin{smallmatrix}n\\ r\end{smallmatrix}\right]_m$ dates back to 1876 when C. Hermite showed that if n is odd and p is an odd prime then $\left[\begin{smallmatrix}n\\ 0\end{smallmatrix}\right]_{p-1}\equiv 1 \pmod p$ (cf. L. E. Dickson [D, p. 271]). In 1899 J. W. L. Glaisher obtained the following generalization of Hermite's result:

$$\begin{bmatrix}n+p-1\\ r\end{bmatrix}_{p-1}\equiv\begin{bmatrix}n\\ r\end{bmatrix}_{p-1} \pmod p \quad \text{for any prime } p.$$

(See, e.g., [Gr, (1.11)].) If p is a prime with $p\equiv 1 \pmod m$, then $\left[\begin{smallmatrix}p\\ r\end{smallmatrix}\right]_m\equiv\left[\begin{smallmatrix}1\\ r\end{smallmatrix}\right]_m \pmod p$ since p divides any of $\binom{p}{1},\ldots,\binom{p}{p-1}$, thus $\left[\begin{smallmatrix}n+p-1\\ r\end{smallmatrix}\right]_m\equiv\left[\begin{smallmatrix}n\\ r\end{smallmatrix}\right]_m \pmod p$ by (1.2) and induction. This explains Glaisher's result in a simple way. (Recently the author and R. Tauraso [ST] obtained a further extension of Glaisher's congruence.) In the modern investigations made by Z. H. Sun and the author (cf. [SS], [S], [Su1] and [Su2]), $\left[\begin{smallmatrix}n\\ r\end{smallmatrix}\right]_m$ was expressed in terms of linear recurrences and then applied to produce congruences for primes. The sum $\left[\begin{smallmatrix}n\\ r\end{smallmatrix}\right]_m$ also appeared in C. Helou's study of Terjanian's conjecture concerning Hilbert's residue symbol and cyclotomic units (cf. [H, Prop. 2 and Lemma 3]).

Now we state two theorems on the sums in (1.1) and give two corollaries. The proofs of them depend heavily on an identity established by the author in [Su3], and will be presented in Section 2.

Theorem 1.1. *Let m be a positive integer. Then, for any integers k and $n\geqslant 2\lfloor (m-1)/2\rfloor$, we have*

$$\sum_{i=0}^{\lfloor (m-1)/2\rfloor}(-1)^i\binom{m-1-i}{i}\begin{bmatrix}n-2i\\ k-i\end{bmatrix}_m=2^{n-m+1}+\delta_{m-2,\,n}\frac{(-1)^k}{2}, \tag{1.3}$$

where the Kronecker symbol $\delta_{l,n}$ is 1 or 0 according to whether $l=n$ or not.

Corollary 1.1. *Let $k\in\mathbb{Z}$ and $m\in\mathbb{Z}^+$. For $n\in\mathbb{N}$ set*

$$u_n=\begin{bmatrix}n\\ \lfloor (k+n)/2\rfloor\end{bmatrix}_m \quad \text{and } v_n=mu_n-2^n-\delta_{n,0}\delta_{(-1)^m,1}(-1)^{\lfloor k/2\rfloor}, \tag{1.4}$$

where $\lfloor \alpha\rfloor$ denotes the integral part of a real number α. Then we have

$$\sum_{i=0}^{\lfloor (m-1)/2\rfloor}(-1)^i\binom{m-1-i}{i}u_{n-2i}=2^{n-m+1}-\delta_{m-2,\,n}\frac{(-1)^{\lfloor (k+m)/2\rfloor}}{2} \tag{1.5}$$

for every integer $n \geqslant 2\lfloor (m-1)/2 \rfloor$. *Also,* $(v_n)_{n\in\mathbb{N}}$ *is a linear recurrence sequence, satisfying the recurrence:*

$$\sum_{i=0}^{\lfloor (m-1)/2\rfloor} (-1)^i \binom{m-1-i}{i} v_{n-2i} = 0 \quad \text{for all } n \geqslant 2\left\lfloor \frac{m-1}{2} \right\rfloor. \tag{1.6}$$

Remark 1.1. (a) In fact, the author first proved (1.6) in the case $2 \nmid m$ on August 1, 1988, motivated by a conjecture of Z. H. Sun; after reading the author's initial proof Z. H. Sun [S] noted that the equality in (1.6) also holds if $2 \mid m$ and $n \geqslant m-1$. (b) In light of the first equality in (1.2), on August 11, 1988 the author obtained the following result by induction: Let $m, n \in \mathbb{Z}^+$ and $m > 2$. If $n \geqslant m-1$ then

$$\begin{bmatrix} n \\ \lfloor \frac{n+1}{2} \rfloor \end{bmatrix}_m > \begin{bmatrix} n \\ \lfloor \frac{n+1}{2} \rfloor + 1 \end{bmatrix}_m > \cdots > \begin{bmatrix} n \\ \lfloor \frac{n+m}{2} \rfloor \end{bmatrix}_m,$$

otherwise

$$\begin{bmatrix} n \\ \lfloor \frac{n+1}{2} \rfloor \end{bmatrix}_m > \cdots > \begin{bmatrix} n \\ n \end{bmatrix}_m > \begin{bmatrix} n \\ n+1 \end{bmatrix}_m = \cdots = \begin{bmatrix} n \\ \lfloor \frac{n+m}{2} \rfloor \end{bmatrix}_m = 0.$$

Therefore

$$\begin{bmatrix} n \\ \lfloor \frac{n}{2} \rfloor \end{bmatrix}_m > \frac{2^n}{m} > \begin{bmatrix} n \\ \lfloor \frac{m+n}{2} \rfloor \end{bmatrix}_m.$$

Theorem 1.2. *Let* $k \in \mathbb{Z}$ *and* $m \in \mathbb{Z}^+$. *Then*

$$\sum_{i=0}^{\lfloor (m+1)/2\rfloor} (-1)^i c_m(i) \begin{bmatrix} n-2i \\ k-i \end{bmatrix}_m = 2(-1)^k \delta_{m,n} \tag{1.7}$$

for each integer $n \geqslant 2\lfloor (m+1)/2 \rfloor$, *and*

$$\sum_{i=0}^{\lfloor m/2\rfloor} (-1)^i d_m(i) \begin{Bmatrix} n-2i \\ k-i \end{Bmatrix}_m = (-1)^k \delta_{m-1,n} \tag{1.8}$$

for any integer $n \geqslant 2\lfloor m/2 \rfloor$, *where* $c_1(1) = 4$, *and*

$$c_m(i) = \frac{m^2+m-2i}{(m-i)(m+1-i)} \binom{m+1-i}{i} \in \mathbb{Z} \text{ and } d_m(i) = \frac{m}{m-i}\binom{m-i}{i} \in \mathbb{Z}$$

for every $i = 0, \ldots, m-1$.

Remark 1.2. Let p be an odd prime. It is easy to check that

$$(-1)^{i-1} c_{p-1}(i) \equiv (-1)^i d_{p-1}(i) \equiv C_i \pmod p \quad \text{for } i = 1, 2, \cdots, \frac{p-1}{2},$$

where $C_i = \binom{2i}{i}/(i+1) = \binom{2i}{i} - \binom{2i}{i+1}$ is the i-th Catalan number.

Corollary 1.2 (A. Fleck, 1913). *Let $n\in\mathbb{Z}^+$ and $r\in\mathbb{Z}$. If p is a prime, then*

$$\sum_{\substack{0\leqslant k\leqslant n\\ p\mid k-r}}(-1)^k\binom{n}{k}\equiv 0 \ \left(\mathrm{mod}\ p^{\lfloor (n-1)/(p-1)\rfloor}\right). \tag{1.9}$$

By [Su2, Remark 2.1], for $k\in\mathbb{Z}$, $l\in\mathbb{N}$, $m\in\mathbb{Z}^+$ and $\varepsilon\in\{1,-1\}$, we have

$$\sum_{\gamma^m=\varepsilon}\gamma^k(2-\gamma-\gamma^{-1})^l=(-1)^k m\times\begin{cases}\left[{2l\atop k+l}\right]_m & \text{if } \varepsilon=(-1)^m,\\ \left\{{2l\atop k+l}\right\}_m & \text{otherwise.}\end{cases}$$

So Theorem 1.2 is closely related to the following materials on Bernoulli and Euler polynomials.

Let $m,n\in\mathbb{Z}^+$, $q\in\mathbb{Z}$ and $(q,m)=1$, where (q,m) is the greatest common divisor of q and m. If $\gamma^m=1$ and $\gamma^{(q,m)}=\gamma\neq 1$, then $\gamma^q\neq 1$ and hence $2-\gamma^q-\gamma^{-q}\neq 0$. We define a linear recurrence $(U_l^{(q)}(m,n))_{l\in\mathbb{N}}$ of order $\lfloor m/2\rfloor$ by

$$U_l^{(q)}(m,n)=\frac{1}{2m}\sum_{\substack{\gamma^m=1\\ \gamma\neq 1}}\frac{2-\gamma^{qn}-\gamma^{-qn}}{2-\gamma^q-\gamma^{-q}}(2-\gamma-\gamma^{-1})^l. \tag{1.10}$$

Note that $U_l^{(-q)}(m,n)=U_l^{(q)}(m,n)$ and

$$mU_l^{(q)}(m,n)=(1-(-1)^{(m-1)n})2^{2l-2}+\sum_{\substack{d\mid m\\ d>2}}u_l^{(q)}(d,n),$$

where

$$\begin{aligned}u_l^{(q)}(d,n)&=\sum_{\substack{0<c<d/2\\ (c,d)=1}}\frac{2-e^{2\pi i\frac{c}{d}qn}-e^{-2\pi i\frac{c}{d}qn}}{2-e^{2\pi i\frac{c}{d}q}-e^{-2\pi i\frac{c}{d}q}}\left(2-e^{2\pi i\frac{c}{d}}-e^{-2\pi i\frac{c}{d}}\right)^l\\ &=\sum_{\substack{0<c<d/2\\ (c,d)=1}}\left(\frac{\sin(\pi nqc/d)}{\sin(\pi qc/d)}\right)^2\left(4\sin^2\frac{\pi c}{d}\right)^l.\end{aligned}$$

Obviously $u_l^{(q)}(m,n)=mU_l^{(q)}(m,n)$ if m is an odd prime. Later we will see that $U_0^{(q)}(m,n)=n(m-n)/(2m)$ if $1\leqslant n\leqslant m$, and $U_l^{(q)}(m,n)\in\mathbb{Z}$ if $l>0$. When $(q,2m)=1$, for $l\in\mathbb{N}$ we also define

$$V_l^{(q)}(m,n)=\frac{1}{2m}\sum_{\gamma^m=-1}\frac{2-\gamma^{qn}-\gamma^{-qn}}{2-\gamma^q-\gamma^{-q}}(2-\gamma-\gamma^{-1})^l; \tag{1.11}$$

clearly $V_l^{(\pm q)}(m,n) = 2U_l^{(q)}(2m,n) - U_l^{(q)}(m,n)$ since $\gamma^m = -1$ if and only if $\gamma^{2m} = 1$ but $\gamma^m \neq 1$.

Let p be an odd prime, and let $m, n > 0$ be integers with $p \nmid m$ and $m \nmid n$. A. Granville and the author [GS, pp. 126–129] proved the following surprising result for Bernoulli polynomials: If $p \equiv \pm q \pmod m$ where $q \in \mathbb{Z}$, then

$$B_{p-1}\left(\left\{\frac{pn}{m}\right\}\right) - B_{p-1} \equiv \frac{m}{2p}\left(U_p^{(q)}(m,n) - 1\right) \pmod p \tag{1.12}$$

where we use $\{\alpha\}$ to denote the fractional part of a real number α. (The reader may consult [Su4] for other congruences concerning Bernoulli polynomials.) With the help of Theorem 1.2, we can write the recurrent coefficients of the sequence $(U_l^{(q)}(m,n))_{l\in\mathbb{N}}$ in a simple closed form.

Theorem 1.3. *Let $m, n \in \mathbb{Z}^+$, $q \in \mathbb{Z}$ and $(q,m) = 1$. Then we have the recursions:*

$$U_l^{(q)}(m,n) = \sum_{0<i\leqslant\lfloor m/2\rfloor} (-1)^{i-1} a_m(i) U_{l-i}^{(q)}(m,n) \quad \text{for } l \geqslant \left\lfloor \frac{m}{2} \right\rfloor, \tag{1.13}$$

and

$$V_l^{(q)}(m,n) = \sum_{j=1}^{\lfloor (m+1)/2\rfloor} (-1)^{j-1} b_m(j) V_{l-j}^{(q)}(m,n) \quad \text{for } l \geqslant \left\lfloor \frac{m+1}{2} \right\rfloor \tag{1.14}$$

provided $(q, 2m) = 1$, where the integers $a_m(i)$ and $b_m(j)$ are given by

$$a_m(i) = \begin{cases} c_m(i) & \text{if } 2 \mid m, \\ d_m(i) & \text{if } 2 \nmid m, \end{cases} \quad \text{and} \quad b_m(j) = \begin{cases} d_m(j) & \text{if } 2 \mid m, \\ c_m(j) & \text{if } 2 \nmid m. \end{cases} \tag{1.15}$$

If m does not divide n, and p is an odd prime with $p \equiv \pm q \pmod{2m}$, then

$$(-1)^{\lfloor pn/m \rfloor} E_{p-2}\left(\left\{\frac{pn}{m}\right\}\right) + \frac{2^p - 2}{p} \equiv \frac{m}{p}\left(V_p^{(q)}(m,n) - 1\right) \pmod p, \tag{1.16}$$

where the Euler polynomials $E_k(x)$ $(k = 0, 1, \dots)$ are given by

$$\frac{2e^{xz}}{e^z + 1} = \sum_{k=0}^{\infty} E_k(x) \frac{z^k}{k!}.$$

In Section 3 we will first deduce Theorem 1.3 from Theorem 1.2, and then give another proof of (1.13) via Chebyshev polynomials. Section 4 is an appendix containing the explicit values of $a_m(i)$ and $b_m(j)$ for $m = 2, 3, \dots, 12$.

2. Proofs of Theorems 1.1 and 1.2

Lemma 2.1. *Let l be any nonnegative integer. Then*

$$\sum_{j=0}^{l}(-1)^{l-j}\binom{x+y+j}{l-j}\binom{y+2j}{j}=\sum_{j=0}^{l}\binom{l-x}{j}. \tag{2.1}$$

Proof. Since both sides of (2.1) are polynomials in x and y, it suffices to show (2.1) for all $x\in\{l,l+1,\dots\}$ and $y\in\{0,2,4,\dots\}$.

Let $x=l+n$ and $y=2k$ where $n,k\in\mathbb{N}$. Set $m=k+l$. Then

$$\begin{aligned}&\sum_{j=0}^{l}(-1)^{l-j}\binom{x+y+j}{l-j}\binom{y+2j}{j}\\ =&\sum_{i=k}^{m}(-1)^{l-(i-k)}\binom{x+2k+i-k}{l-(i-k)}\binom{2k+2(i-k)}{i-k}\\ =&(-1)^m\sum_{i=k}^{m}(-1)^i\binom{m+n+i}{m-i}\binom{2i}{k+i}\\ =&\sum_{j=0}^{l}(-1)^j\binom{n+j-1}{j}\quad\text{(by [Su3, (3.2)])}\\ =&\sum_{j=0}^{l}\binom{-n}{j}=\sum_{j=0}^{l}\binom{l-x}{j}.\end{aligned}$$

This concludes the proof. □

Remark 2.1. Lemma 2.1, an equivalent version of [Su3, (3.2)], played a key role when the author established the following curious identity in [Su3]:

$$\begin{aligned}&(x+m+1)\sum_{i=0}^{m}(-1)^i\binom{x+y+i}{m-i}\binom{y+2i}{i}\\ &=\sum_{i=0}^{m}\binom{x+i}{m-i}(-4)^i+(x-m)\binom{x}{m},\end{aligned} \tag{2.2}$$

where m is any nonnegative integer. The reader is referred to [C], [CC], [EM], [MS] and [PP] for other proofs of (2.2), and to [SW] for an extension of (2.2). In the case $x\in\{0,\dots,l\}$ the right-hand side of (2.1) turns out to be 2^{l-x}, so (2.1) implies identity (3) in [C], which has a nice combinatorial interpretation.

Proof of Theorem 1.1. Let $n\in\mathbb{Z}$ and $n\geqslant 2h$, where $h=\lfloor(m-1)/2\rfloor$. Then $n+1\geqslant m-1>m-2$. Suppose that (1.3) holds for all $k\in\mathbb{Z}$. Then, for any given

$k \in \mathbb{Z}$, we have

$$\begin{aligned}
&\sum_{i=0}^{h}(-1)^i\binom{m-1-i}{i}\begin{bmatrix}n+1-2i\\ k-i\end{bmatrix}_m\\
=&\sum_{i=0}^{h}(-1)^i\binom{m-1-i}{i}\left(\begin{bmatrix}n-2i\\ k-i\end{bmatrix}_m+\begin{bmatrix}n-2i\\ k-1-i\end{bmatrix}_m\right)\\
=&\sum_{i=0}^{h}(-1)^i\binom{m-1-i}{i}\begin{bmatrix}n-2i\\ k-i\end{bmatrix}_m+\sum_{i=0}^{h}(-1)^i\binom{m-1-i}{i}\begin{bmatrix}n-2i\\ k-1-i\end{bmatrix}_m\\
=&2^{n-m+1}+\delta_{m-2,\,n}\frac{(-1)^k}{2}+\left(2^{n-m+1}+\delta_{m-2,\,n}\frac{(-1)^{k-1}}{2}\right)=2^{(n+1)-m+1}.
\end{aligned}$$

In view of the above, it suffices to show (1.3) for $n=2h$ and $k\in\{0,1,\dots,m-1\}$. For any $i\in\mathbb{N}$ with $i\leqslant h$, we have $k-i+m>n-2i$ since $n-m<0\leqslant k+i$, thus

$$\begin{bmatrix}n-2i\\ k-i\end{bmatrix}_m=\begin{cases}\binom{n-2i}{k-i} & \text{if } i\leqslant k,\\ 0 & \text{if } i>k.\end{cases}$$

Let $x=m-1-n+k$, $y=n-2k$, and Σ denote the left hand side of (1.3). Then

$$\begin{aligned}
\Sigma=&\sum_{i=0}^{k}(-1)^i\binom{m-1-i}{i}\binom{n-2i}{k-i}\\
=&\sum_{j=0}^{k}(-1)^{k-j}\binom{x+y+j}{k-j}\binom{y+2j}{j}\\
=&\sum_{j=0}^{k}\binom{k-x}{j}=\sum_{j=0}^{k}\binom{n-(m-1)}{j}
\end{aligned}$$

with the help of Lemma 2.1. If m is odd, then $n=m-1$ and hence $\Sigma=\sum_{j=0}^{k}\binom{0}{j}=1=2^{n-m+1}$. If m is even, then $n=m-2$ and

$$\Sigma=\sum_{j=0}^{k}\binom{-1}{j}=\sum_{j=0}^{k}(-1)^j=\frac{1+(-1)^k}{2}=2^{n-m+1}+\frac{(-1)^k}{2}.$$

So we do have $\Sigma=2^{n-m+1}+\delta_{m-2,\,n}(-1)^k/2$ as required. □

Proof of Corollary 1.1. Let $n\in\mathbb{N}$ and $n\geqslant 2\lfloor(m-1)/2\rfloor$. By Theorem 1.1,

$$\sum_{i=0}^{\lfloor(m-1)/2\rfloor}(-1)^i\binom{m-1-i}{i}\begin{bmatrix}n-2i\\ \lfloor\frac{k+n}{2}\rfloor-i\end{bmatrix}_m=2^{n-m+1}+\delta_{m-2,\,n}\frac{(-1)^{\lfloor(k+n)/2\rfloor}}{2}.$$

If $m-2=n$, then $2\mid m$ and $(k+n)/2=(k+m)/2-1$. So (1.5) holds.

For $0\leqslant i\leqslant 2\lfloor (m-1)/2\rfloor$, if $n-2i=0$ and $2\mid m$, then we must have $n/2=i=\lfloor (m-1)/2\rfloor = m/2-1$. Note also that

$$\sum_{i=0}^{\lfloor (m-1)/2\rfloor}(-1)^i\binom{m-1-i}{i}2^{m-1-2i}=m$$

by (1.60) of [G] or (4) of [C]. Therefore

$$\begin{aligned}&\sum_{i=0}^{\lfloor (m-1)/2\rfloor}(-1)^i\binom{m-1-i}{i}v_{n-2i}\\ =&m\sum_{i=0}^{\lfloor (m-1)/2\rfloor}(-1)^i\binom{m-1-i}{i}u_{n-2i}-\sum_{i=0}^{\lfloor (m-1)/2\rfloor}(-1)^i\binom{m-1-i}{i}2^{n-2i}\\ &-\sum_{i=0}^{\lfloor (m-1)/2\rfloor}(-1)^i\binom{m-1-i}{i}\delta_{n-2i,\,0}\delta_{(-1)^m,1}(-1)^{\lfloor k/2\rfloor}\\ =&-m\delta_{m-2,\,n}\frac{(-1)^{\lfloor (k+m)/2\rfloor}}{2}-\delta_{m-2,\,n}(-1)^{\lfloor (m-1)/2\rfloor}\frac{m}{2}(-1)^{\lfloor k/2\rfloor}=0.\end{aligned}$$

This concludes the proof. $\square$

Proof of Theorem 1.2. i) Clearly $c_m(0)=1$. As $\lfloor m/2\rfloor+\lfloor (m+1)/2\rfloor=m$, whether $m=1$ or not, we have

$$c_m\left(\left\lfloor\frac{m+1}{2}\right\rfloor\right)=4\binom{\lfloor\frac{m}{2}\rfloor}{\lfloor\frac{m-1}{2}\rfloor}=4\binom{m-\lfloor\frac{m+1}{2}\rfloor}{\lfloor\frac{m+1}{2}\rfloor-1}.$$

If $0<i<m/2$ then

$$\begin{aligned}c_m(i)=&\frac{(m-i)!}{i!(m-2i)!}\cdot\frac{m^2+m-2i}{(m-i)(m+1-2i)}=\frac{(m-i)!}{i!(m-2i)!}\left(\frac{m-2i}{m-i}+\frac{4i}{m+1-2i}\right)\\ =&\frac{(m-1-i)!}{i!(m-1-2i)!}+4\frac{(m-i)!}{(i-1)!(m+1-2i)!}=\binom{m-1-i}{i}+4\binom{m-i}{i-1}.\end{aligned}$$

Let $n\in\mathbb{N}$ and $n\geqslant 2\lfloor (m+1)/2\rfloor$. Set $h=\lfloor (m-1)/2\rfloor$. As $n>n-2\geqslant 2h$, by Theorem 1.1 we have

$$\sum_{i=0}^{h}(-1)^i\binom{m-1-i}{i}\begin{bmatrix}n-2i\\ k-i\end{bmatrix}_m=2^{n-m+1}$$

and

$$\sum_{i=0}^{h}(-1)^i\binom{m-1-i}{i}\begin{bmatrix}n-2-2i\\ k-1-i\end{bmatrix}_m=2^{n-2-m+1}+\delta_{m,n}\frac{(-1)^{k-1}}{2}.$$

Therefore

$$\begin{aligned}0 =&2^{n-m+1}-4\cdot 2^{n-2-m+1}\\ =&\sum_{i=0}^{h}(-1)^i\binom{m-1-i}{i}\begin{bmatrix}n-2i\\ k-i\end{bmatrix}_m\\ &-4\bigg(\sum_{i=0}^{h}(-1)^i\binom{m-1-i}{i}\begin{bmatrix}n-2-2i\\ k-1-i\end{bmatrix}_m+\delta_{m,n}\frac{(-1)^k}{2}\bigg)\end{aligned}$$

and hence

$$\begin{aligned}2(-1)^k\delta_{m,n} =&\begin{bmatrix}n\\ k\end{bmatrix}_m+\sum_{0<i<m/2}(-1)^i\binom{m-1-i}{i}\begin{bmatrix}n-2i\\ k-i\end{bmatrix}_m\\ &+4\sum_{j=1}^{h+1}(-1)^j\binom{m-j}{j-1}\begin{bmatrix}n-2j\\ k-j\end{bmatrix}_m\\ =&\begin{bmatrix}n\\ k\end{bmatrix}_m+\sum_{0<i<m/2}(-1)^i\left(\binom{m-1-i}{i}+4\binom{m-i}{i-1}\right)\begin{bmatrix}n-2i\\ k-i\end{bmatrix}_m\\ &+(-1)^{\lfloor\frac{m+1}{2}\rfloor}4\binom{m-\lfloor\frac{m+1}{2}\rfloor}{\lfloor\frac{m+1}{2}\rfloor-1}\begin{bmatrix}n-2\lfloor\frac{m+1}{2}\rfloor\\ k-\lfloor\frac{m+1}{2}\rfloor\end{bmatrix}_m\\ =&\sum_{i=0}^{h+1}(-1)^i c_m(i)\begin{bmatrix}n-2i\\ k-i\end{bmatrix}_m.\end{aligned}$$

This proves the first part of Theorem 1.2.

ii) Observe that

$$\frac{m}{m-i}\binom{m-i}{i}=2\binom{m-i}{i}-\binom{m-1-i}{i}\in\mathbb{Z}\quad\text{for } i=0,\ldots,m-1.$$

In view of (1.2), it suffices to verify (1.8) in the case $n=2\lfloor m/2\rfloor$ and $0\leqslant k<m$. For any $i\in\mathbb{N}$ with $i\leqslant n/2=\lfloor m/2\rfloor$, we have $k-i+m>n-2i$ if and only if $i=k=0$ and $m=n$, and thus

$$\begin{Bmatrix}n-2i\\ k-i\end{Bmatrix}_m=\begin{cases}0 & \text{if } i>k,\ \text{or } i=k=0\ \&\ m=n,\\ \binom{n-2i}{k-i} & \text{otherwise.}\end{cases}$$

Therefore

$$\begin{aligned}
&\sum_{i=0}^{\lfloor m/2\rfloor}(-1)^i d_m(i)\begin{Bmatrix}n-2i\\ k-i\end{Bmatrix}_m\\
=&\sum_{i=0}^{k}(-1)^i\frac{m}{m-i}\binom{m-i}{i}\binom{n-2i}{k-i}-\delta_{k,0}\delta_{m,n}\\
=&2\sum_{i=0}^{k}(-1)^i\binom{m-i}{i}\binom{n-2i}{k-i}-\sum_{i=0}^{k}(-1)^i\binom{m-1-i}{i}\binom{n-2i}{k-i}-\delta_{k,0}\delta_{m,n}\\
=&2\sum_{j=0}^{k}\binom{n-m}{j}-\sum_{j=0}^{k}\binom{n-(m-1)}{j}-\delta_{k,0}\delta_{m,n}=\delta_{m-1,\,n}(-1)^k
\end{aligned}$$

with the help of Lemma 2.1.

The proof of Theorem 1.2 is now complete. □

Proof of Corollary 1.2. The case $p=2$ can be verified directly, so let $p>2$. Clearly, (1.9) holds if and only if $p^{\lfloor (n-1)/(p-1)\rfloor}\mid \left\{{n\atop r}\right\}_p$. If $n\geqslant p$, then $\left\{{n\atop r}\right\}_p=\sum_{i=1}^{\lfloor p/2\rfloor}(-1)^{i-1}d_p(i)\left\{{n-2i\atop r-i}\right\}_p$ by Theorem 1.2. Since $p\mid d_p(i)$ for $i=1,\ldots,\lfloor p/2\rfloor$, we have the desired result by induction on n. □

3. Proof of Theorem 1.3

Let $m\in\mathbb{Z}^+$, $n\in\mathbb{N}$ and $r\in\mathbb{Z}$. Set

$$(3.1)\qquad \binom{n}{r}_m=\begin{cases}\left[{n\atop r}\right]_m & \text{if } 2\mid m,\\ \left\{{n\atop r}\right\}_m & \text{if } 2\nmid m;\end{cases}\quad\text{and}\quad \binom{n}{r}_m^*=\begin{cases}\left\{{n\atop r}\right\}_m & \text{if } 2\mid m,\\ \left[{n\atop r}\right]_m & \text{if } 2\nmid m.\end{cases}$$

Clearly

$$(-1)^r\binom{n}{r}_m=\sum_{\substack{k=0\\ m\mid k-r}}^{n}\binom{n}{k}(-1)^k,\quad (-1)^r\binom{n}{r}_m^*=\sum_{\substack{k=0\\ m\mid k-r}}^{n}\binom{n}{k}(-1)^{k+(k-r)/m}$$

and

$$\binom{n}{r}_m+\binom{n}{r}_m^*=\begin{bmatrix}n\\ r\end{bmatrix}_m+\begin{Bmatrix}n\\ r\end{Bmatrix}_m=2\begin{bmatrix}n\\ r\end{bmatrix}_{2m}=2\binom{n}{r}_{2m}.$$

Since $\left[{n\atop n-r}\right]_m=\left[{n\atop r}\right]_m$ and $\left\{{n\atop n-r}\right\}_m=\left\{{n\atop r}\right\}_m$, we have $\binom{n}{n-r}_m=\binom{n}{r}_m$ and also $\binom{n}{n-r}_m^*=\binom{n}{r}_m^*$.

Lemma 3.1. *Let $l \in \mathbb{N}$, $m, n \in \mathbb{Z}^+$, $q \in \mathbb{Z}$ and $(q, m) = 1$. Then*

$$U_l^{(q)}(m,n) = \sum_{r=0}^{n} \frac{n-r}{1+\delta_{r,0}} \left((-1)^{qr} \binom{2l}{l+qr}_m - \frac{\delta_{l,0}}{m} \right), \tag{3.2}$$

and $U_l^{(q)}(m,n) \in \mathbb{Z}$ if $1 \leqslant l \leqslant \lfloor (m+1)/2 \rfloor$. When $(q, 2m) = 1$, we have

$$V_l^{(q)}(m,n) = \sum_{r=0}^{n} \frac{n-r}{1+\delta_{r,0}} (-1)^{qr} \binom{2l}{l+qr}_m^*, \tag{3.3}$$

and also $V_l^{(q)}(m,n) \in \mathbb{Z}$ provided $1 \leqslant l \leqslant \lfloor (m+1)/2 \rfloor$.

Proof. Let $\rho = 1$, or $\rho = -1$ and $(q, 2m) = 1$. With the help of the identity

$$\frac{2 - x^n - x^{-n}}{2 - x - x^{-1}} = n + \sum_{r=1}^{n} (n-r)(x^r + x^{-r}) = \sum_{r=-n}^{n} (n - |r|) x^r$$

(cf. [GS, (2.2)], we have

$$\begin{aligned}
&\sum_{\substack{\gamma^m = \rho \\ \gamma \neq 1}} \frac{2 - \gamma^{qn} - \gamma^{-qn}}{2 - \gamma^q - \gamma^{-q}} (2 - \gamma - \gamma^{-1})^l \\
&= \sum_{\substack{\gamma^m = \rho \\ \gamma \neq 1}} \sum_{r=-n}^{n} (n - |r|) \gamma^{qr} (2 - \gamma - \gamma^{-1})^l \\
&= \sum_{r=-n}^{n} (n - |r|) \left(\sum_{\gamma^m = \rho} \gamma^{qr} (2 - \gamma - \gamma^{-1})^l - \delta_{\rho,1} \delta_{l,0} \right) \\
&= \sum_{r=-n}^{n} (n - |r|) \times \begin{cases} (-1)^{qr} m \binom{2l}{l+qr}_m - \delta_{l,0} & \text{if } \rho = 1, \\ (-1)^{qr} m \binom{2l}{l+qr}_m^* & \text{if } \rho = -1, \end{cases}
\end{aligned}$$

where in the last step we apply Remark 2.1 of [Su2]. Note that $\binom{2l}{l+qr}_m = \binom{2l}{l-qr}_m$ and $\binom{2l}{l+qr}_m^* = \binom{2l}{l-qr}_m^*$. So we have (3.2), also (3.3) holds if $(q, 2m) = 1$.

Suppose that $1 \leqslant l \leqslant \lfloor (m+1)/2 \rfloor$. In the case $m = 1$, both $\binom{2l}{l}_m = 0$ and $\binom{2l}{l}_m^* = 4$ are even. If $m > 1$, then $l + m > 2l$ and hence $\binom{2l}{l}_m = \binom{2l}{l}_m^* = \binom{2l}{l} = 2\binom{2l-1}{l}$. Therefore $U_l^{(q)}(m,n) \in \mathbb{Z}$, and also $V_l^{(q)}(m,n) \in \mathbb{Z}$ when $(q, 2m) = 1$.

The proof of Lemma 3.1 is now complete. □

Remark 3.1. Let $q \in \mathbb{Z}$, $m \in \mathbb{Z}^+$ and $(q, m) = 1$. In view of (3.2), we have

$$U_0^{(q)}(m,n) = \frac{n}{2} \left(1 - \frac{1}{m} \right) = \frac{n(m-n)}{2m} \quad \text{for } n = 1, \ldots, m. \tag{3.4}$$

If $m > 1$, then

$$U_l^{(q)}(m,1) = \frac{1}{2}\binom{2l}{l} = \binom{2l-1}{l} \quad \text{for } l = 1, \cdots, \left\lfloor \frac{m+1}{2} \right\rfloor. \tag{3.5}$$

When $(q, 2m) = 1$, $V_0^{(q)}(m,n) = 2U_0^{(q)}(2m,n) - U_0^{(q)}(m,n) = n/2$ for $n = 1, \ldots, m$, and also $V_l^{(q)}(m,1) = 2U_l^{(q)}(2m,1) - U_l^{(q)}(m,1) = \binom{2l-1}{l}$ if $m > 1$ and $1 \leqslant l \leqslant \lfloor (m+1)/2 \rfloor$.

For positive integers m and n, it is known that

$$\sum_{r=0}^{m-1} B_n\left(\frac{x+r}{m}\right) = m^{1-n} B_n(x) \tag{3.6}$$

(due to Raabe), and

$$E_{n-1}(x) = \frac{2}{n}\left(B_n(x) - 2^n B_n\left(\frac{x}{2}\right)\right).$$

Lemma 3.2. *Let n be a positive integer, and let x be a real number. Then*

$$nE_{n-1}(\{x\}) = 2(-1)^{\lfloor x \rfloor}\left(B_n(\{x\}) - 2^n B_n\left(\left\{\frac{x}{2}\right\}\right)\right).$$

Proof. Clearly $2\{x/2\} - \{x\} = \lfloor x \rfloor - 2\lfloor x/2 \rfloor \in \{0, 1\}$. If $2 \mid \lfloor x \rfloor$, then

$$B_n(\{x\}) - 2^n B_n\left(\left\{\frac{x}{2}\right\}\right) = B_n(\{x\}) - 2^n B_n\left(\frac{\{x\}}{2}\right) = \frac{n}{2} E_{n-1}(\{x\}).$$

By Raabe's formula (3.6),

$$B_n\left(\frac{\{x\}}{2}\right) + B_n\left(\frac{\{x\}+1}{2}\right) = 2^{1-n} B_n(\{x\}).$$

So, if $2 \nmid \lfloor x \rfloor$ then

$$\begin{aligned} B_n(\{x\}) - 2^n B_n\left(\left\{\frac{x}{2}\right\}\right) =& B_n(\{x\}) - 2^n B_n\left(\frac{\{x\}+1}{2}\right) \\ =& B_n(\{x\}) - 2^n\left(2^{1-n} B_n(\{x\}) - B_n\left(\frac{\{x\}}{2}\right)\right) \\ =& 2^n B_n\left(\frac{\{x\}}{2}\right) - B_n(\{x\}) = -\frac{n}{2} E_{n-1}(\{x\}). \end{aligned}$$

This concludes the proof. $\square$

From Lemma 3.2 we have

Lemma 3.3. *Let p be an odd prime, and let $m, n \in \mathbb{Z}^+$ and $p \nmid m$. Then*

$$\begin{aligned}&\frac{(-1)^{\lfloor pn/m \rfloor}}{2} E_{p-2}\left(\left\{\frac{pn}{m}\right\}\right) + \frac{2^{p-1}-1}{p} \\ \equiv & B_{p-1}\left(\left\{\frac{pn}{2m}\right\}\right) - B_{p-1} - \left(B_{p-1}\left(\left\{\frac{pn}{m}\right\}\right) - B_{p-1}\right) \pmod p.\end{aligned} \tag{3.7}$$

Proof. By Lemma 3.2,

$$\begin{aligned}&(-1)^{\lfloor pn/m \rfloor}\frac{p-1}{2} E_{p-2}\left(\left\{\frac{pn}{m}\right\}\right) + (2^{p-1}-1)B_{p-1} \\ =& B_{p-1}\left(\left\{\frac{pn}{m}\right\}\right) - B_{p-1} - 2^{p-1}\left(B_{p-1}\left(\left\{\frac{pn}{2m}\right\}\right) - B_{p-1}\right)\end{aligned}$$

As $2^{p-1} \equiv 1 \pmod p$ by Fermat's little theorem, and $pB_{p-1} \equiv -1 \pmod p$ by [IR, p. 233], the desired (3.7) follows at once. $\square$

Remark 3.2. Let p be an odd prime not dividing $m \in \mathbb{Z}^+$. By [GS, pp. 125–126] or [Su5, Corollary 2.1],

$$B_{p-1}\left(\left\{\frac{pn}{m}\right\}\right) - B_{p-1} \equiv -\sum_{k=1}^{\lfloor pn/m \rfloor} \frac{1}{k} \pmod p \quad \text{for } n = 0, \ldots, m-1.$$

Combining this with (3.7) we get that

$$\begin{aligned}&\frac{(-1)^{\lfloor pn/m \rfloor}}{2} E_{p-2}\left(\left\{\frac{pn}{m}\right\}\right) + \frac{2^{p-1}-1}{p} \\ \equiv & \sum_{k=1}^{\lfloor pn/m \rfloor} \frac{1}{k} - \sum_{k=1}^{\lfloor pn/(2m) \rfloor} \frac{1}{k} = \sum_{k=1}^{\lfloor pn/m \rfloor} \frac{(-1)^{k-1}}{k} \pmod p\end{aligned}$$

for every $n = 0, \ldots, m-1$. In light of Lemma 3.3, we can also deduce from (3) and (4) of [GS] the following congruences with $n \in \mathbb{Z}^+$ and $(m, n) = 1$.

$$(-1)^{\lfloor pn/m \rfloor} E_{p-2}\left(\left\{\frac{pn}{m}\right\}\right) \equiv \begin{cases} (\frac{2}{n})\frac{4}{p}P_{p-(\frac{2}{p})} \pmod p & \text{if } m = 4, \\ (\frac{n}{5})\frac{5}{p}F_{p-(\frac{5}{p})} + \frac{2^p-2}{p} \pmod p & \text{if } m = 5, \\ (\frac{3}{pn})\frac{6}{p}S_{p-(\frac{3}{p})} \pmod p & \text{if } m = 6, \end{cases} \tag{3.8}$$

where $(-)$ denotes the Jacobi symbol, and the sequences $(F_k)_{k\in\mathbb{N}}$, $(P_k)_{k\in\mathbb{N}}$ and $(S_k)_{k\in\mathbb{N}}$ are defined as follows:

$$\begin{aligned}&F_0 = 0,\ F_1 = 1,\ \text{and } F_{k+2} = F_{k+1} + F_k \text{ for } k \in \mathbb{N}; \\ &P_0 = 0,\ P_1 = 1,\ \text{and } P_{k+2} = 2P_{k+1} + P_k \text{ for } k \in \mathbb{N}; \\ &S_0 = 0,\ S_1 = 1,\ \text{and } S_{k+2} = 4S_{k+1} - S_k \text{ for } k \in \mathbb{N}.\end{aligned}$$

Proof of Theorem 1.3. Let $k \in \mathbb{Z}$. By Theorem 1.2, for any integer $l \geqslant \lfloor m/2 \rfloor$ we have

$$\begin{aligned}&\sum_{i=0}^{\lfloor m/2\rfloor}(-1)^i a_m(i)\bigg((-1)^k\binom{2l-2i}{k+l-i}_m-\frac{\delta_{l-i,\,0}}{m}\bigg)\\ =&(1+\delta_{(-1)^m,1})(-1)^l\delta_{l,\lfloor m/2\rfloor}-\frac{\delta_{l,\lfloor m/2\rfloor}}{m}(-1)^l a_m\left(\left\lfloor\frac{m}{2}\right\rfloor\right)=0;\end{aligned}$$

also

$$\sum_{j=0}^{\lfloor (m+1)/2\rfloor}(-1)^j b_m(j)\binom{2l-2j}{k+l-j}_m^*=0$$

for all integers $l \geqslant \lfloor (m+1)/2 \rfloor$. This, together with Lemma 3.1, yields (1.13), and also (1.14) in the case $(q, 2m) = 1$.

By Lemma 3.1, $U_l^{(q)}(m,n) \in \mathbb{Z}$ for every $l = 1, \ldots, \lfloor (m+1)/2 \rfloor$; by Theorem 1.2, $a_m(i) \in \mathbb{Z}$ if $0 < i \leqslant \lfloor m/2 \rfloor$. Thus, in view of (1.13), we have $U_l^{(q)}(m,n) \in \mathbb{Z}$ for each $l = 1, 2, 3, \ldots$. If $(q, 2m) = 1$, then $V_l^{(q)}(m,n) = 2U_l^{(q)}(2m,n) - U_l^{(q)}(m,n) \in \mathbb{Z}$ for all $l \in \mathbb{Z}^+$.

Now assume that $m \nmid n$, and let p be an odd prime with $p \equiv \pm q \pmod{2m}$. By Lemma 3.3 and (1.12),

$$\begin{aligned}&(-1)^{\lfloor pn/m\rfloor}E_{p-2}\left(\left\{\frac{pn}{m}\right\}\right)+\frac{2^p-2}{p}\\ \equiv&2\bigg(B_{p-1}\left(\left\{\frac{pn}{2m}\right\}\right)-B_{p-1}\bigg)-2\bigg(B_{p-1}\left(\left\{\frac{pn}{m}\right\}\right)-B_{p-1}\bigg)\\ \equiv&\frac{2m}{p}\left(U_p^{(q)}(2m,n)-1\right)-\frac{m}{p}\left(U_p^{(q)}(m,n)-1\right)=\frac{m}{p}\left(V_p^{(q)}(m,n)-1\right)\pmod p.\end{aligned}$$

This proves (1.16). We are done. $\square$

We can also prove (1.13) by determining the characteristic polynomial

$$f_m(x):=\prod_{0<k\leqslant\lfloor m/2\rfloor}\Big(x-\Big(2-e^{2\pi ik/m}-e^{-2\pi ik/m}\Big)\Big)\tag{3.9}$$

of the recurrence $(U_l^{(q)}(m,n))_{l\in\mathbb{N}}$ of order $\lfloor m/2 \rfloor$. If m is even, then

$$\begin{aligned}f_m(x)=&\prod_{0<k\leqslant m/2}\Big(x-2-e^{2\pi i(m/2-k)/m}-e^{-2\pi i(m/2-k)/m}\Big)\\ =&\prod_{j=0}^{m/2-1}\bigg(x-2-2\cos\frac{2j\pi}{m}\bigg)=(x-4)\prod_{\substack{0<k<m\\ 2\mid k}}\bigg(x-4\cos^2\frac{k\pi}{2m}\bigg).\end{aligned}$$

If m is odd, then

$$\begin{aligned} f_m(x) &= \prod_{0<j\leqslant(m-1)/2}\left(x-2-e^{2\pi i(m-2j)/(2m)}-e^{-2\pi i(m-2j)/(2m)}\right)\\ &= \prod_{\substack{0<k<m\\ 2\nmid k}}\left(x-2-2\cos\frac{2k\pi}{2m}\right) = \prod_{\substack{0<k<m\\ 2\nmid k}}\left(x-4\cos^2\frac{k\pi}{2m}\right). \end{aligned}$$

So $f_m(x)$ can be determined with the help of the following lemma.

Lemma 3.4. *Let n be any positive integer. Then*

$$\prod_{\substack{0<k<n\\ 2\mid k-\delta}}\left(x-4\cos^2\frac{k\pi}{2n}\right)=\begin{cases} C_n(x) & \text{if } \delta=0,\\ D_n(x) & \text{if } \delta=1,\end{cases} \tag{3.10}$$

where

$$C_n(x)=\sum_{i=0}^{\lfloor (n-1)/2\rfloor}(-1)^i\binom{n-1-i}{i}x^{\lfloor (n-1)/2\rfloor-i} \tag{3.11}$$

and

$$D_n(x)=\sum_{i=0}^{\lfloor n/2\rfloor}(-1)^i\frac{n}{n-i}\binom{n-i}{i}x^{\lfloor n/2\rfloor-i}. \tag{3.12}$$

Proof. It is well known that $\cos(n\theta)=T_n(\cos\theta)$ and $\sin(n\theta)=\sin\theta\cdot U_{n-1}(\cos\theta)$, where the Chebyshev polynomials $T_n(x)$ and $U_{n-1}(x)$ are given by

$$T_n(x)=\frac{n}{2}\sum_{i=0}^{\lfloor n/2\rfloor}(-1)^i\frac{(n-1-i)!}{i!(n-2i)!}(2x)^{n-2i}$$

and

$$U_{n-1}(x)=\sum_{i=0}^{\lfloor (n-1)/2\rfloor}(-1)^i\frac{(n-1-i)!}{i!(n-1-2i)!}(2x)^{n-1-2i}.$$

If n is even, then $T_n(x)=D_n(4x^2)/2$ and $U_{n-1}(x)=2xC_n(4x^2)$; if n is odd, then $T_n(x)=xD_n(4x^2)$ and $U_{n-1}(x)=C_n(4x^2)$.

As $U_{n-1}(\cos\frac{k\pi}{2n})=0$ for those even $0<k<n$, the $2\lfloor (n-1)/2\rfloor$ distinct numbers $\pm\cos\frac{k\pi}{2n}$ ($0<k<n$, $2\mid k$) are zeroes of the polynomial $C_n(4x^2)$ of degree $2\lfloor (n-1)/2\rfloor$. Similarly, since $T_n(\cos\frac{k\pi}{2n})=0$ for those odd $0<k<n$, the $2\lfloor n/2\rfloor$

distinct numbers $\pm\cos\frac{k\pi}{2n}$ $(0<k<n,\ 2\nmid k)$ are zeroes of the polynomial $D_n(4x^2)$ of degree $2\lfloor n/2\rfloor$. So

$$C_n(4x^2)=\prod_{\substack{0<k<n\\ 2\mid k}}\left(4x^2-4\cos^2\frac{k\pi}{2n}\right)\ \text{and}\ D_n(4x^2)=\prod_{\substack{0<k<n\\ 2\nmid k}}\left(4x^2-4\cos^2\frac{k\pi}{2n}\right).$$

Therefore (3.10) holds. □

Remark 3.3. For each $n\in\mathbb{Z}^+$, by Lemma 3.4 we have

$$C_n(x)=\prod_{0<k<n/2}\left(x-4\cos^2\frac{k\pi}{n}\right)=\prod_{d\mid n}A_d(x),$$

where

$$A_d(x)=\prod_{\substack{0<c<d/2\\ (c,d)=1}}\left(x-4\cos^2\frac{c\pi}{d}\right). \tag{3.13}$$

Applying the Möbius inversion formula we obtain

$$A_n(x)=\prod_{d\mid n}C_d(x)^{\mu(n/d)}, \tag{3.14}$$

which makes the polynomial $A_n(x)$ (introduced in [Su2]) computable.

4. Appendix: Explicit Values of $a_m(i)$ and $b_m(j)$ for $2\leqslant m\leqslant 12$

Table 1: Values of $a_m(i)$ with $2\leqslant m\leqslant 12$

m \ i	1	2	3	4	5	6
2	4					
3	3					
4	6	8				
5	5	5				
6	8	19	12			
7	7	14	7			
8	10	34	44	16		
9	9	27	30	9		
10	12	53	104	85	20	
11	11	44	77	55	11	
12	14	76	200	259	146	24

Table 2: Values of $b_m(j)$ with $2 \leqslant m \leqslant 12$

$m \backslash j$	1	2	3	4	5	6
2	2					
3	5	4				
4	4	2				
5	7	13	4			
6	6	9	2			
7	9	26	25	4		
8	8	20	16	2		
9	11	43	70	41	4	
10	10	35	50	25	2	
11	13	64	147	155	61	4
12	12	54	112	105	36	2

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